Multimonopoles and closed vortices in $SU(2)$ Yang-Mills-Higgs theory

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Abstract

We review classical monopole solutions of the $SU(2)$ Yang-Mills-Higgs theory. The first part is a pedagogical introduction into to the basic features of the celebrated 't Hooft - Polyakov monopole. In the second part we describe new classes of static axially symmetric solutions which generalise 't Hooft - Polyakov monopole. These configurations are either deformations of the topologically trivial sector or the sectors with different topological charges. In both situations we construct the solutions representing the chains of monopoles and antimonopoles in static equilibrium. The solutions of another type are closed vortices which are centred around the symmetry axis and form different bound systems. Configurations of the third type are monopoles bounded with vortices. We suggest classification of these solutions which is related with $2d$ Poincare index.

Introduction

There are two completely distinct periods in the long history of the monopole problem. Soon after the pioneering work by Dirac [1], the interest in this field of research faded away although some enthusiasts still worked on it [1]. Actually, for more then 40 years the monopole problem was considered to be a rather esoteric, fairly beyond mainstream of theoretical physics of that time. Such an attitude was caused in part by negative results of all the experiments searching for monopoles, and it was reinforced by rather cumbersome character of the theoretical constructions which were connected with the corresponding generalisation of quantum electrodynamics. As a matter of fact this problem still remains unsolved and, within an Abelian theory, a monopole looks like a stranger. Indeed, a magnetic monopole could be introduced into the Abelian electrodynamics on the classical level if a vector potential is not defined globally or if there are singular objects in the theory (for related discussion see e.g. [2]). However, even in that case the quantum theory of the monopole is full of contradictions one can hardly avoid in the framework of an Abelian model.

The situation changes drastically if we take into account that the Abelian electrodynamics is not a lonely theory but a part of an unified model, i.e., the generator of the electromagnetic $U(1)$ subgroup has to be embedded into a non-Abelian gauge group of higher rank.

The modern era of the monopole theory started in 1974 when 't Hooft and Polyakov independently discovered monopole solutions of the $SO(3)$ Georgi–Glashow model [5, 6]. Preskill pointed out that the essence of this breakthrough is that while a Dirac monopole *could be* incorporated in an Abelian theory, some non-Abelian models, like that of Georgi and Glashow, *inevitably contain* monopole-like solutions [11].

For many years, starting from the pioneering paper by Dirac, the most serious argument to support the monopole concept, apart from his emotional belief that "one would be surprised if Nature had made no use of it" [1], was the possible explanation of the quantisation of the electric charge. But as time went on and the idea of grand unification emerged, it seemed that the latter argument has lost some power. Indeed, the modern point of view is that the operator of electric charge is the generator of a $U(1)$ group. The charge quantisation condition arises in models of unification if the electromagnetic subgroup is embedded into a semi-simple non-Abelian gauge group of higher rank. In this case the electric charge generator forms nontrivial commutation relations with all other generators of the gauge group. Therefore, the electric charge quantisation today is considered as an argument in support of the unification approach.

However it turns out that both the 'old' and 'new' explanations of the electric charge quantisation are just two sides of the same problem, because it was realised that almost any theory of unification with an electromagnetic $U(1)$ subgroup embedded into a gauge

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[1] For the complete discussion of 'stone age history' of the monopole problem, the properties of the Dirac monopole and exhaustive bibliography before 1974 see [2, 3, 4]. Note that in [4] the genesis of the monopole problem was traced up to the notes by Petrus Pelegrinius, written at the Crusades in 1269! We will not go into this fascinating story.

group, which is spontaneously broken by the Higgs mechanism, possesses monopole-like solutions. An example of such a model is the $SU(2)$ Yang-Mills-Higgs (YMH) theory which we shall consider below.

1 't Hooft - Polyakov monopole

1.1 $SU(2)$ Yang-Mills-Higgs model

We consider non-Abelian classical Lagrangian which describes coupled gauge and Higgs fields:

$$
\begin{aligned}
L &= -\frac{1}{2}\mathrm{Tr}F_{\mu\nu}F^{\mu\nu} + \mathrm{Tr}D_\mu\phi D^\mu\phi - V(\phi) \\
&= -\frac{1}{4}F_{\mu\nu}^a F^{a\mu\nu} + \frac{1}{2}(D^\mu\phi^a)(D_\mu\phi^a) - V(\phi) \ .
\end{aligned}
\tag{1}
$$

Here, $F_{\mu\nu} = F_{\mu\nu}^a T^a$, $\phi = \phi^a T^a$ and we use standard normalisation of the generators of the gauge group: $\mathrm{Tr}(T^a T^b) = \frac{1}{2}\delta_{ab}$, $a,b = 1,2,3$, which satisfy the Lie algebra

$$
[T^a, T^b] = i\varepsilon_{abc}T^c \ .
\tag{2}
$$

In what follows we will choose the simplest non-trivial case of $SU(2)$ group [2], thus the generators could be taken in the fundamental representation, $T^a = \frac{\sigma^a}{2}$ or in the adjoint representation, $(T^a)_{bc} = -i\varepsilon_{abc}$ respectively. The covariant derivative is defined as

$$
D_\mu = \partial_\mu + ieA_\mu \ ,
\tag{3}
$$

which yields

$$
D_\mu\phi = \partial_\mu\phi + ie[A_\mu, \phi], \quad \text{or} \quad D_\mu\phi^a = \partial_\mu\phi^a - e\varepsilon_{abc}A_\mu^b\phi^c \ ,
\tag{4}
$$

and the potential of the scalar fields is taken to be

$$
V(\phi) = \frac{\lambda}{4}(\phi^a\phi^a - a^2)^2 \ ,
\tag{5}
$$

where e and λ are gauge and scalar coupling constants respectively. The field strength tensor is

$$
F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a - e\varepsilon_{abc}A_\mu^b A_\nu^c \ ,
\tag{6}
$$

or in matrix form

$$
F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ie[A_\mu, A_\nu] \equiv \frac{1}{ie}[D_\mu, D_\nu] \ .
\tag{7}
$$

---

[2]Note that the simply connected covering $SU(2)$ group is locally isomorphic to the $SO(3)$. In the monopole theory the global difference between these two groups is very important because the topological properties of the corresponding group spaces are different. For the sake of brevity we do not discuss this point and refer the reader to [11].

The field equations corresponding to the Lagrangian (1)

$$D_\nu F^{a\mu\nu} = -e\varepsilon_{abc}\phi^b D^\mu \phi^c \; ; \qquad D_\mu D^\mu \phi^a = -\lambda \phi^a (\phi^b \phi^b - a^2) \; . \tag{8}$$

The symmetric stress-energy tensor $T_{\mu\nu}$ which follows from the Lagrangian (1) and the field equations (8) is

$$
\begin{aligned}
T_{\mu\nu} &= F^a_{\mu\rho} F^{a\rho}_\nu + (D_\mu \phi^a)(D_\nu \phi^a) - g_{\mu\nu} L \\
&= F^a_{\mu\alpha} F^{\nu\alpha a} + D_\mu \phi^a D_\nu \phi^a - \frac{1}{2} g_{\mu\nu} D_\alpha \phi^a D^\alpha \phi^a - \frac{1}{4} g_{\mu\nu} F^a_{\alpha\beta} F^{\alpha\beta a} \\
&\quad - g_{\mu\nu} \frac{\lambda}{4} \left(\phi^2 - a^2 \right) \; ,
\end{aligned}
\tag{9}
$$

and is conserved by virtue of the field equations: $\partial_\mu T^{\mu\nu} = 0$. From (9) we can easily obtain the static Hamiltonian

$$
\begin{aligned}
E &= \int d^3 x \, T_{00} = \int d^3 x \left[\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu a} + \frac{1}{2} (D_\mu \phi^a)(D^\mu \phi^a) + \frac{\lambda}{4} (\phi^a \phi^a - a^2)^2 \right] \\
&= \int d^3 x \frac{1}{2} \left[E^a_n E^a_n + B^a_n B^a_n + (D_n \phi^a)(D_n \phi^a) \right] + V(\phi) \; ,
\end{aligned}
\tag{10}
$$

where

$$E^a_n \equiv F^a_{0n}, \qquad \text{and} \qquad B^a_n \equiv \frac{1}{2} \varepsilon_{nmk} F^a_{mk} \tag{11}$$

are 'colour' electric and magnetic fields. We see that the energy is minimal if the following conditions are satisfied

$$\phi^a \phi^a = a^2; \qquad F^a_{mn} = 0; \qquad D_n \phi^a = 0 \; . \tag{12}$$

These conditions define the vacuum.

The perturbative spectrum of the model can be found from analysing small fluctuations around the vacuum state. Let us suppose that the system under consideration is static, $E^a_n = 0$. Then the energy of the vacuum is equal to zero. We consider now a fluctuation χ of the scalar field ϕ around the trivial vacuum $|\phi| = a$ where only the third isotopic component of the Higgs field is non-vanishing:

$$\phi = (0, 0, a + \chi) \; . \tag{13}$$

Substitution of the expansion (13) into the Lagrangian (1) yields, up to terms of the second order

$$(D_n \phi^a)(D_n \phi^a) \approx (\partial_n \chi^a)(\partial_n \chi^a) + e^2 a^2 \left[\left(A^1_n \right)^2 + \left(A^2_n \right)^2 \right] \; , \tag{14}$$

and

$$V(\phi) \approx \frac{\lambda}{2} a^2 \chi^2 \; . \tag{15}$$

Thus, the vacuum average of the scalar field is non-vanishing and the model describes spontaneous symmetry breaking. Further analysis shows that the perturbative spectrum

consists of a massless photon A_μ^3 corresponding to the unbroken $U(1)$ electromagnetic subgroup, massive vector fields $A_\mu^\pm = (1/\sqrt{2})(A_\mu^1 \pm A_\mu^2)$ with mass $m_v = ea$, and neutral scalars having a mass $m_s = a\sqrt{2\lambda}$.

Note that the electric charge of the massive vector bosons $A^\pm$ is given by the unbroken $U(1)$ subgroup. In general, this is a subgroup H of the gauge group G, the action of which leaves the Higgs vacuum invariant. Obviously, that is a little group of the rotation in isospace about the direction given by the vector ϕ^a. The generator of it, $(\phi^a T^a)/a$, must be identified with the operator of electric charge Q. Thus, the expression for the covariant derivative (4) can be written in the form

$$D_\mu = \partial_\mu + ieA_\mu^a T^a = \partial_\mu + iQA_\mu^{\text{em}} \tag{16}$$

that allows to define the 'electromagnetic projection' of the gauge potential

$$A_\mu^{\text{em}} = \frac{1}{a}A_\mu^a \phi^a, \qquad Q = e\frac{1}{a}\phi^a T^a. \tag{17}$$

Taking into account the definition of the generators T^a of the gauge group, we can easily see that the minimal allowed eigenvalues of the electric charge operator now are $q = \pm e/2$.

1.2 Topological classification of the solutions

The spectrum of possible solutions of the Yang-Mills-Higgs model is much richer than one would naively expect. There are stable soliton-like static solutions of the complicated system of field equations (8) having a finite energy density on the spatial asymptotic. An adequate description of these objects needs the extensive use of topological methods.

Indeed, the very definition (12) forces the classical vacuum of the Yang-Mills-Higgs theory to be degenerated. The condition $V(\phi) = 0$ means that $|\phi| = a$, i.e., the set of vacuum values of the Higgs field forms a sphere S_{vac}^2 of radius a in $d = 3$ isotopic space. All the points on this sphere are equivalent because there is a well defined $SU(2)$ gauge transformation which connects them.

The solutions of the classical field equations map the vacuum manifold $\mathcal{M} = S_{\text{vac}}^2$ onto the boundary of 3-dimensional space, which is also a sphere S^2. These maps are charactered by a *winding number* $n = 0, \pm 1, \pm 2 \ldots$ which is the number of times S_{vac}^2 is covered by a single turn around the spatial boundary S^2. The crucial point is that the solutions having a finite energy on the spatial asymptotic could be separated into different classes according to the behaviour of the field ϕ^a. The trivial case is that the isotopic orientation of the fields do not depend on the spatial coordinates and asymptotically the scalar field tends to the limit

$$\phi^a = (0, 0, a) \tag{18}$$

This situation corresponds to winding number $n = 0$.

We can also consider another type of solutions with the property that the direction of isovector and isoscalar fields in isospace are functions of the spatial coordinates. One could suppose that since the absolute minimum of the energy corresponds to the trivial vacuum,

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such configurations would be unstable. However, the stability of them will be secured by the topology, if we try to deform the fields continuously to the trivial vacuum (18), then the energy functional would tend to infinity. In other words, all the different topological sectors are separated by infinite barriers.

To construct the solutions corresponding to the non-trivial minimum of the energy functional (10), we again consider the scalar field on the spatial asymptotic $r \to \infty$ taking values on the vacuum manifold $|\phi| = a$. However, we suppose that the isovector of the scalar field now is directed in the isotopic space along the direction of the radius vector on the spatial asymptotic[3]

$$\phi^a \stackrel{r \to \infty}{\longrightarrow} \frac{ar^a}{r} \tag{19}$$

This asymptotic behaviour obviously mixes the spatial and isotopic indices and defines a single mapping of the vacuum $\mathcal{M}$ onto the spatial asymptotic, a single turn around the boundary S^2 leads to a single closed path on the sphere S^2_{vac} and the winding number of such a mapping is $n = 1$.

As was mentioned by 't Hooft [5], the configurations which are characterised by different winding numbers cannot be continuously deformed into each other. Indeed, the gauge transformation of the form $U = e^{i(\sigma_k \hat{\varphi}_k)\theta/2}$ rotates the isovector to the third axis. However, if we try to 'comb the hedgehog', that is, to rotate the scalar field everywhere in space to a given direction (so called *unitary* or *Abelian gauge*), the singularity of the gauge transformation on the south pole does not allow to do it globally. Thus, there is no well-defined global gauge transformation which connects the configurations (18) and (19) and this singularity results in the infinite barrier separating them.

1.3 Definition of magnetic charge

The condition of vanishing covariant derivative of the scalar field on the spatial asymptotic (12) together with the choice of the nontrivial hedgehog configuration implies that at $r \to \infty$

$$\partial_n \left(\frac{r^a}{r} \right) - e\varepsilon_{abc} A_n^b \frac{r^c}{r} = 0. \tag{20}$$

The simple transformation

$$\partial_n \left(\frac{r^a}{r} \right) = \frac{r^2 \delta_{an} - r_a r_n}{r^3} = \frac{1}{r} \left(\delta_{an}\delta_{ck} - \delta_{ak}\delta_{nc} \right) \frac{r_c r_k}{r^2} = -\varepsilon_{abc}\varepsilon_{bnk} \frac{r_c r_k}{r^3}$$

then provides an asymptotic form of the gauge potential

$$A_k^a(r) \stackrel{r \to \infty}{\longrightarrow} \frac{1}{e} \varepsilon_{ank} \frac{r_n}{r^2} \tag{21}$$

This corresponds to the non-Abelian magnetic field

$$B_n^a \stackrel{r \to \infty}{\longrightarrow} \frac{r_a r_n}{er^4} \tag{22}$$

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[3]In the pioneering paper by Polyakov [6] this solution was coined a 'hedgehog'.

Therefore, the boundary conditions (19), (21) are compatible with the existence of a long-range gauge field associated with an Abelian subgroup which is unbroken in the vacuum. Since this field falls off like $1/r^2$, which characterises the Coulomb-like field of a point charge, and since the electric components of the field strength tensor (6) vanish, this configuration with a finite energy could be identified with a monopole.

To prove it, we first have to define the electromagnetic field strength tensor. Recall that the unbroken electromagnetic subgroup $U(1)$ is associated with rotations about the direction of the isovector ϕ. Thus it would be rather natural to introduce the electromagnetic potential as a projection of the $SU(2)$ gauge potential A_μ^a onto that direction, see eq. (17). Furthermore, as was mentioned in the paper [7], a general solution of the equation $D_\mu \phi^a = 0$, for $\phi^a \phi^a = a^2$, can be written as

$$A_\mu^a = \frac{1}{a^2 e}\varepsilon_{abc}\phi^b \partial_\mu \phi^c + \frac{1}{a}\phi^a \Lambda_\mu \tag{23}$$

where Λ_μ is an arbitrary 4-vector. It can be identified with the electromagnetic potential because eq. (23) yields for $\phi^a \phi^a = a^2$:

$$\frac{\phi^a}{a}A_\mu^a = \Lambda_\mu \equiv A_\mu^{\mathrm{em}}.$$

Inserting eq. (23) into the definition of the field strength tensor (6) yields

$$F_{\mu\nu}^a = F_{\mu\nu}\frac{\phi^a}{a}, \qquad \text{where} \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + \frac{1}{a^3 e}\varepsilon_{abc}\phi^a \partial_\mu \phi^b \partial_\nu \phi^c \tag{24}$$

This gauge-invariant definition of the electromagnetic field strength tensor $F_{\mu\nu}$ suggested by 't Hooft [5], has a very deep meaning. It is rather obvious that in the topologically trivial sector (18) the last term in eq. (24) vanishes and then we have

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

This is precisely the case of standard Maxwell electrodynamics. Of course, in this sector there is no place for a monopole because the Bianchi identities are satisfied: $\partial^\mu \widetilde{F}_{\mu\nu} \equiv 0$.

However, for the configuration with non-trivial boundary conditions (19), (21), also the Higgs field gives a non-vanishing contribution to the electromagnetic field strength tensor (24). Then, the second pair of Maxwell equations becomes modified:

$$\partial^\mu \widetilde{F}_{\mu\nu} = k_\nu \tag{25}$$

Note, that if the electromagnetic potential A_μ^{em} is regular, the magnetic, or topological current k_μ is expressed via the scalar field alone

$$k_\mu = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}\partial^\nu F^{\rho\sigma} = \frac{1}{2a^3 e}\varepsilon_{\mu\nu\rho\sigma}\varepsilon_{abc}\partial^\nu \phi^a \partial^\rho \phi^b \partial^\sigma \phi^c. \tag{26}$$

From the first glance this current is independent of any property of the gauge field. It is conserved by its very definition:

$$\partial_\mu k_\mu \equiv 0 \ , \tag{27}$$

unlike a Noether current which is conserved because of some symmetry.

Now we can justify the definition of magnetic charge [8]. According to eq. (26)

$$
\begin{aligned}
g &= \int d^3x k_0 = \frac{1}{2ea^3} \int d^3x \, \varepsilon_{abc}\varepsilon_{mnk} \, \partial_m \left(\phi^a \partial_n \phi^b \partial_k \phi^c \right) \\
&= -\frac{1}{2ea^3} \int d^2 S_n \, \varepsilon_{abc}\varepsilon_{mnk} \, \phi^a \partial_n \phi^b \partial_k \phi^c \ ,
\end{aligned}
\tag{28}
$$

where the last integral is taken over the surface of the sphere S^2 on the spatial asymptotic. One can parameterise it by local coordinates ξ_α, $\alpha = 1, 2$. Then we can write

$$
\partial_n \phi^a = \frac{\partial \xi^\alpha}{\partial r^n} \frac{\partial \phi^a}{\partial \xi^\alpha}; \qquad d^2 S_n = \frac{1}{2} \, \varepsilon_{nmk} \frac{\partial r^m}{\partial \xi^\alpha} \frac{\partial r^k}{\partial \xi^\beta} \varepsilon_{\alpha\beta} d^2 \xi.
\tag{29}
$$

After some simple algebra we arrive at

$$
g = \frac{1}{2ea^3} \int d^2 \xi \, \varepsilon_{\alpha\beta}\varepsilon_{abc} \, \phi^a \partial_\alpha \phi^b \partial_\beta \phi^c = \frac{1}{e} \int d^2 \xi \sqrt{\mathbf{g}} \ ,
\tag{30}
$$

where $\mathbf{g} = \det(\partial_\alpha \hat{\phi}^a \partial_\beta \hat{\phi}^a)$ is the determinant of the metric tensor on the S^2_{vac} sphere in isospace. The magnetic charge is proportional to an integer n which mathematicians refer to as the Brouwer degree. The geometrical interpretation of this integer is clear: it is the number of times the isovector ϕ^a covers the sphere S^2_{vac} while r^a covers the sphere S^2 on the spatial asymptotic once. Thus [8]:

$$
g = \frac{4\pi n}{e}, \qquad n \in \mathbb{Z} \ ,
\tag{31}
$$

where the factor 4π is due to integration over the unit sphere. This is the non-Abelian counterpart of the Dirac charge quantisation condition.

Another remark about the definition of the magnetic charge is that the Brouwer degree and homotopic classification are equivalent to the 3d Poincaré–Hopf index [8]. The latter is defined as a mapping of a sphere S^2 surrounding an isolated point r_0 where the scalar field vanishes, i.e., $\phi(r_0) = 0$, onto a sphere of unit radius S^2_ϕ (see e.g. [63]). In other words, the magnetic charge of an arbitrary field configuration can be defined as a sum of the Poincaré–Hopf indices i of non-degenerated zeros $r_0^{(k)}$ of the Higgs field:

$$
g = \frac{4\pi}{e} \sum_k i(r_0^{(k)}) \ .
\tag{32}
$$

Indeed, if we consider a scalar field which is constant everywhere in space and satisfies the boundary condition (18), it has no zero at all. Thus, the 3d Poincaré–Hopf index, alias magnetic charge, is equal to zero. However, in the case of the hedgehog configuration which satisfies the boundary condition (19)

$$
\phi^a = r^a h(r) \ ,
\tag{33}
$$

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where $h(r)$ is a smooth function having no zeros, there is a single zero at the origin. Thus, $i(0) = 1$ and that is a configuration of unit magnetic charge.

This approach allows one to identify monopoles according to the positions of zeros of the Higgs field. Such an identification is very useful from the point of view of constructing multi-monopole solutions which we will consider in the following section. But first we have to find a solution of the field equations (8), which would satisfy the boundary conditions (19) and (21).

1.4 't Hooft–Polyakov ansatz

We showed that asymptotically, the monopole field configuration must satisfy the conditions (19) and (21). Now, we try to define the structure functions which form the radial shape of the monopole. As usual, this problem can be simplified if we take into account the constraints following from the symmetries of the configuration. Note, that we consider static fields. That condition leaves only rotational $SO(3)$ symmetry from the original Poincaré invariance of the Lagrangian (1). Therefore, the full invariance group of the system is $SO(3) \times SU(2)$, the product of spatial and group rotations. Moreover, the non-trivial asymptotic of the Higgs field (19) corresponds to the symmetry with respect to the transformation from the diagonal $SO(3)$ subgroup which mix spatial and group rotations. Thus, one can make the ansatz [5, 6]:

$$\phi^a = \frac{r^a}{er^2}H(aer); \quad A_n^a = \varepsilon_{amn}\frac{r^m}{er^2}[1 - K(aer)]; \quad A_0^a = 0 ,$$

(34)

where $H(aer)$ and $K(aer)$ are functions of the dimensionless variable $\xi = aer$. The explicit forms of these shape functions of the scalar and gauge field can be found from the field equations. However, it would be much more convenient to make use of the condition that the monopole solution corresponds to a local minimum of the energy functional. Substituting the ansatz (34) back into eq. (10) we have

$$
E = \frac{4\pi a}{e} \int_0^\infty \frac{d\xi}{\xi^2} \left[\xi^2 \left(\frac{dK}{d\xi}\right)^2 + \frac{1}{2}\left(\xi\frac{dH}{d\xi} - H\right)^2 \right.
$$
$$
\left. + \frac{1}{2}\left(K^2 - 1\right)^2 + K^2 H^2 + \frac{\lambda}{4e^2}\left(H^2 - \xi^2\right)^2 \right] .
$$

Variations of this functional with respect to the functions H and K yields

$$\xi^2\frac{d^2K}{d\xi^2} = KH^2 + K(K^2 - 1); \quad \xi^2\frac{d^2H}{d\xi^2} = 2K^2H + \frac{\lambda}{e^2}H(H^2 - \xi^2) .$$

(35)

The functions K and H must satisfy the following boundary conditions:

$$
\begin{aligned}
K(\xi) &\to 1, & H(\xi) &\to 0 \quad \text{for} \quad \xi \to 0 ; \\
K(\xi) &\to 0, & H(\xi) &\to \xi \quad \text{for} \quad \xi \to \infty ,
\end{aligned}
$$

(36)

which correspond to the asymptotic (19) and (21). Indeed, the substitution of the ansatz (34) into the expressions for the covariant derivative of the scalar field and the non-Abelian magnetic field yields

$$
\begin{aligned}
D_n\phi^a &= \frac{\delta_{an}}{er^2}KH + \frac{r^a r^n}{er^4}\left(\xi\frac{dH}{d\xi} - H - KH\right) \xrightarrow{r\to\infty} 0; \\
B_n^a &= \frac{r_n r^a}{er^4}\left(1 - K^2 + \xi\frac{dK}{d\xi}\right) - \frac{\delta_{an}}{er^2}\xi\frac{dK}{d\xi} \xrightarrow{r\to\infty} \frac{r_n r^a}{er^4} \; ..
\end{aligned}
\tag{37}
$$

Let us note that in the Higgs vacuum $D_n\phi^a = 0$ and the electromagnetic field strength is $F_{\mu\nu} = \phi^a F_{\mu\nu}^a/a$. Clearly, the magnetic charge could be calculated as an integral over the surface of the sphere S^2 on spatial infinity (compare with eq. (28)):

$$
g = \frac{1}{a}\int d^2 S_n B_n = \frac{1}{a}\int d^2 S_n B_n^a \phi^a = \frac{1}{a}\int d^3x\, B_n^a D_n\phi^a \;,
\tag{38}
$$

where we made use of the Bianchi identity for the tensor of non-Abelian magnetic field $D_n B_n^a = 0$. Substituting the ansatz (34) we obtain

$$
\begin{aligned}
g &= \frac{4\pi}{e}\int_0^\infty \frac{d\xi}{\xi^2}\left\{(K^2 - 1)(H - \xi H') - 2\xi K' K H\right\} \\
&= \frac{4\pi}{e}\int_0^\infty d\xi\frac{d}{d\xi}\left\{\frac{1 - K^2}{\xi}\right\} = \frac{4\pi}{e} \;.
\end{aligned}
\tag{39}
$$

Again we see that the boundary conditions (36) correspond to a monopole of unit magnetic charge.

Numerical solutions of the system (35) were discussed in the papers [9, 10]. It turns out that the shape functions $H(\xi)$ and $K(\xi)$ approach rather fast to the asymptotic values. Thus, there is a Higgs vacuum outside of some region of the order of the characteristic scale R_c, which is called the *core* of the monopole. One could estimate this size by simple arguments [11]. The total energy of the monopole configuration consists of two components: the energy of the Abelian magnetic field outside the core and the energy of the scalar field inside the core:

$$
E = E_{\text{mag}} + E_s \sim 4\pi g^2 R_c^{-1} + 4\pi a^2 R_c \sim \frac{4\pi}{e^2}\left(R_c^{-1} + m_v^2 R_c\right) \;.
$$

This sum is minimal if $R_c \sim m_v^{-1}$. In other words, inside the core at distances shorter than the wavelength of the vector boson $m_v^{-1} \sim (ae)^{-1}$, the original $SU(2)$ symmetry is restored. However, outside the core this symmetry is spontaneously broken down to the Abelian electromagnetic subgroup. In this sense there is no difference between the 't Hooft–Polyakov and the Dirac monopole outside the core [12].

Unfortunately, the system of non-linear coupled differential equations (35) in general has no analytical solution. The only known exception is the very special case $\lambda = 0$ [13, 14, 15]. This is the so-called *Bogomol'nyi–Prasad–Sommerfield (BPS) limit* which deserves a special consideration.

1.5 BPS monopole

In the BPS limit of vanishing Higgs potential the scalar field also becomes massless and the energy of the static field configuration is taking the form

$$E = \int \left\{ \frac{1}{4}\text{Tr}\left((\varepsilon_{ijk}F_{ij} \pm D_k\Phi)^2\right) \mp \frac{1}{2}\varepsilon_{ijk}\text{Tr}\left(F_{ij}D_k\Phi\right)\right\} d^3r \ . \tag{40}$$

Thus, the absolute minimum of the energy corresponds to the static configurations which satisfy the first order Bogomol'nyi equations [13, 14, 15]:

$$\varepsilon_{ijk}F_{ij} = \pm D_k\Phi \tag{41}$$

Substitution of the 't Hooft–Polyakov ansatz (34) yields the system of coupled differential equations of first order

$$\xi\frac{dK}{d\xi} = -KH; \qquad \xi\frac{dH}{d\xi} = H + (1 - K^2) \ , \tag{42}$$

which have an analytical solution in terms of elementary functions:

$$K = \frac{\xi}{\sinh\xi}; \qquad H = \xi\coth\xi - 1 \ . \tag{43}$$

Definitely, the solution to the first order BPS equation (41) automatically satisfies the system of field equations of the second order, (8).

As was mentioned in [16], the BPS equation together with the Bianchi identity means that $D_n D_n\phi^a = 0$, which precisely corresponds to the field equation (8). Therefore, the condition

$$D_n\phi^a D_n\phi^a = (\partial_n\phi^a)(\partial_n\phi^a) + \phi^a(\partial_n\partial_n\phi^a) = \frac{1}{2}\partial_n\partial_n(\phi^a\phi^a)$$

holds. The energy of the monopole configuration in the BPS limit is independent from the properties of the gauge field and completely defined by the Higgs field alone: [16, 17]:

$$E = \frac{1}{2}\int d^3x\,\partial_n\partial_n(\phi^a\phi^a) = \frac{4\pi a}{e}\int\limits_0^\infty d\xi\frac{d}{d\xi}\left[\xi H\frac{d}{d\xi}\left(\frac{H}{\xi}\right)\right]$$

$$= \frac{4\pi a}{e}\left(\coth\xi - \frac{1}{\xi}\right)\left(1 - \frac{\xi^2}{\sinh^2\xi}\right)\Big|_0^\infty = \frac{4\pi a}{e} \ .$$

In comparison with the 't Hooft–Polyakov solution, the behaviour of the Higgs field of the monopole in the BPS limit has changed drastically. As we can see from (43), alongside with the exponentially decaying component it also obtains a long-distance Coulomb tail

$$\phi^a \rightarrow a\hat{r}^a - \frac{r^a}{er^2} \quad \text{for} \quad r \rightarrow \infty \tag{44}$$

The reason for this is that in the limit $V(\phi) = 0$ the scalar field becomes massless. Because an interaction, which is mediated by a massless scalar field, always leads to attraction, the picture of the interaction between the monopoles is very different in the BPS limit, as compared with the naive picture based on pure electromagnetic interaction. Manton showed that monopole–monopole interaction is composed of two parts originated from the long-range scalar force and the standard electromagnetic interaction, which could be either attractive or repulsive [19]. Mutual compensation of both contributions leaves the pair of BPS monopoles static but the monopole and anti-monopole interact with double strength.

Many of the remarkable properties of the BPS equation (41) are connected with its property of integrability. As was pointed out by Manton [22], integrability of the BPS system is connected with a one-to-one correspondence between the system of BPS equations and the reduced equations of self-duality of the pure Euclidean Yang–Mills theory. Indeed, the Julia–Zee correspondence means that

$$
\begin{aligned}
D_n \phi^a &\rightleftharpoons D_n A_0^a \equiv F_{0n}^a, \\
B_n^a = D_n \phi^a &\rightleftharpoons \widetilde{F}_{0n} = F_{0n}^a \ .
\end{aligned}
\tag{45}
$$

Therefore, if we suppose that all the fields are static, the Euclidean equations of self-duality $F_{\mu\nu}^a = \widetilde{F}_{\mu\nu}^a$ reduces to the equations (41) and the monopole solutions in the Bogomol'nyi limit could be considered as a special class of self-dual fields.

Of course, it would not be quite correct to make a direct identification between these fields and instantons, because the instanton configuration could be independent from Euclidean time only in the limit of infinite action. Nevertheless, this analogy opens a way to apply in the $d = 3 + 1$ monopole theory the same very powerful methods of algebraic and differential geometry which were used to construct multi-instanton solutions of the self-duality equations in $d = 4$ [23]. In particular, in the case of the BPS monopole, the solution of the self-duality equations could be constructed on the ansatz of Corrigan and Fairlie [24].

The analogy between the Euclidean Yang–Mills theory and the BPS equations could be traced up to the solutions. It was shown [25, 26], that the solutions of these equations exactly equal to an infinite chain of instantons directed along the Euclidean time axis t in $d = 4$.

It has also been shown by Manton [22], that such a multi-instanton configuration can be written on the 't Hooft ansatz with the help of a superpotential $\rho(r, t)$ as:

$$
A_n^a = \varepsilon_{anm} \partial_m \ln \rho + \delta_{an} \partial_0 \ln \rho; \qquad A_0^a = -\partial_a \ln \rho \ ,
\tag{46}
$$

where the sum over the infinite number of instantons is performed in the superpotential:

$$
\rho = \sum_{n=-\infty}^{n=\infty} \frac{1}{\xi^2 + (\tau - 2\pi n)^2}, \quad \text{where} \quad \xi = aer, \quad \tau = aet \ .
$$

Here, the distance between the neighbouring instantons is equal to 2π in units of τ and the size of the instanton is equal to one in units of ξ.

Rossi pointed out [25] that this sum over instantons could be calculated analytically. Indeed, the superpotential can be decomposed into two sums over Matsubara frequencies $\omega_n = 2\pi n$ which are well known from statistical physics:

$$\rho = \frac{1}{2\xi} \left\{ \sum_{n=-\infty}^{n=\infty} \frac{1}{\xi + i\tau - 2i\pi n} + \sum_{n=-\infty}^{n=\infty} \frac{1}{\xi - i\tau + 2i\pi n} \right\} \tag{47}$$

Introducing the complex variable $z = \xi + i\tau$, we can write

$$\begin{aligned}
\rho &= \frac{1}{2\xi} \left\{ \sum_{n=-\infty}^{n=\infty} \frac{1}{z - i\omega_n} + \sum_{n=-\infty}^{n=\infty} \frac{1}{z^* + i\omega_n} \right\} = \frac{1}{2\xi} \left\{ \coth \frac{z}{2} + \coth \frac{z^*}{2} \right\} \\
&= \frac{1}{2\xi} \frac{\sinh \xi}{\cosh \xi - \cos \tau} \ .
\end{aligned} \tag{48}$$

Substitution of this result into the potential (46) corresponds to the "dyon in the 't Hooft gauge". This solution is periodic in time. However, the time-dependent periodic gauge transformation of the form [22]

$$U = \exp \left\{ \frac{i}{2a} \hat{r}^a \sigma^a \omega \right\}, \qquad \text{where} \quad \tan \omega = \frac{\sin \tau \sinh \xi}{\cosh \xi \cos \tau - 1} \tag{49}$$

transforms the infinite chain of instantons (46) into the form:

$$A_n^a = \varepsilon_{anm} \frac{r^m}{er^2} \left(1 - \frac{\xi}{\sinh \xi} \right); \qquad A_0^a = a\hat{r}^a \left(\coth \xi - \frac{1}{\xi} \right) \ . \tag{50}$$

This is exactly the monopole solution of the BPS equation (43) but with the time component of the gauge potential replacing the scalar field. This is the so called "dyon in the Rossi gauge". Thus, the Julia–Zee correspondence establishes an exact relation between a single BPS monopole and an infinite instanton chain.

As mentioned above, the action of the infinite number of instantons is divergent:

$$S = \sum_n S_1 = \sum_n \frac{8\pi^2 n}{e^2} \to \infty \ . \tag{51}$$

However, the mass of the monopole being defined as an action per unit of Euclidean time is, of course, finite [25]:

$$\frac{dS}{dt} = \frac{8\pi^2}{e^2} \frac{ae}{2\pi} = \frac{4\pi a}{e} \equiv M \ . \tag{52}$$

To sum up, the BPS monopole is equivalent to an infinite chain of instantons having identical orientation in isospace and separated by an interval $\tau_0 = 2\pi$. An alternative configuration is a chain of correlated instanton–anti-instanton pairs, which corresponds to an infinite monopole loop.

2 Multimonopoles

So far, we have considered a single static monopole which has topological charge $n = 1$. An obvious generalisation would be a solution of the field equation with an arbitrary integer topological charge n. At this stage we have to consider two possibilities: a single 'fat' (possibly unstable!) monopole, having a charge $n > 1$, or a system of several monopoles with a total charge equal to n. Obviously, the second situation could be much more interesting, because in this case one would encounter the effects of interaction between the monopoles, their scattering and decay.

Over the last 20 years, the investigation of the exact multi-monopole configurations was definitely at the crossing of the most fascinating directions of modern field theory and differential geometry. This kind of research may have caused more enthusiasm from the side of the mathematicians, rather than the physicists. The point is that the Bogomol'nyi equation is 3-dimensional reduction of the integrable self-duality equations. Thus, its solution is simpler than the investigation of the multi-instanton configurations which arise in Yang–Mills theory in $d = 4$. On the other hand, there is a significant interest to the BPS configurations in supersymmetric theories and relation of these objects to electro-magnetic duality.

The property of integrability makes it possible to find the complete set of multi-monopole solutions. This, however, requires sophisticated mathematical techniques, for example the Atiyah–Drinfeld–Hitchin–Manin (ADHM) construction and other methods developed over the last years. Of course, any attempt to give a detailed description of this fast developing and very intriguing subject is outside the scope of this brief review. We consider here only some elementary aspects of the multi-monopoles.

2.1 BSP multimonopoles: bird's eye view

The exact cancellation of the electromagnetic attraction and the dilaton repulsion in the two-monopole system suggests to conjecture that there are multi-monopole static solutions of the Bogomol'nyi equations. Since Bogomol'nyi has found that the explicit spherically symmetric solution with $n = 1$ is unique [13], any possible multi-monopole configuration with $n > 1$ cannot have such a symmetry. Thus, the structure of the configuration we have to deal with, is rather complicated and it is difficult to obtain these solutions.

Naively, one can visualise the following geometric transformation which could help to construct a multi-monopole configuration starting from a given single spherically symmetric BPS monopole. Recall that magnetic charge is associated with the asymptotic behaviour of the scalar field which form a sphere S^2 for a monopole of charge $n = 1$. To construct a 2-monopole configuration we shall remove from this sphere the equatorial circle S^1 as shown in figure 1, and then identify all the points on the equators of the two hemispheres with the north and south poles of two new spheres respectively. Construction of an n-monopole configuration requires a simple iteration of this procedure[4]. However,

---

[4]With some imagination one can compare this picture with the well known process of biological cell division...
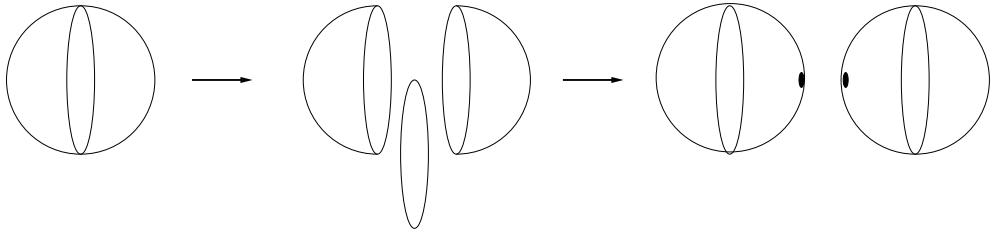
Figure 1: Geometry of the transformation which allows to construct two separated monopoles starting from spherically symmetric one-monopole configuration.

an adequate mathematical description of the geometrical transformation which could solve the multi-monopole problem is not so trivial and emerged only as result of heavy work of many mathematicians lasted over a decade. The problem is not to construct an explicit solution for an arbitrary n-monopole configuration but to prove the completeness of the solution, i.e., to prove that all possible solutions are generated by this procedure.

To give some clue which methods we have to use, let us note that the naive picture above is closely connected with the mathematical apparatus of projective geometry. Indeed, making a standard stereographic projection of the sphere S^2 onto a plane, we see that removing the equator of the sphere corresponds to a cut on the projective plane which then becomes isomorphic to a doubly covered Riemann surface. Recall also that identification of the antipodal points of a sphere S^n transform it into a real projective space $\mathbb{R}P^n$. Thus, it is no surprise that the powerful methods of twistor geometry were fully exploited to construct the multi-monopole configurations.

Generally speaking, to construct a general explicit n-monopole solution, one has to make use of the integrability of the Bogomol'nyi equation. There are three different approaches to this problem which use:

1. twistor technique (the so called Atiyah–Ward ansatz [33, 36, 16, 37]);

2. Atiyah–Drinfeld–Hitchin–Manin (ADHM) construction [34], which was modified by Nahm [35];

3. inverse scattering method (Riemann-Hilbert problem), which was applied to the linearised Bogomol'nyi equation [30, 31, 32].

The reader wishing to know more about first and second directions should consult the classic book by Atiyah and Hitchin [23] and the original papers [27, 28, 16]. A comprehensive review of the mathematical aspects of this problem can be found in [29]. The very detailed review by Nahm in the collection [38] is essential reading. Recent developments are discussed in a very good review by Sutcliffe [17]. For a detailed description of the last approach we refer the reader to the comprehensive review [32]. Here, we briefly outline another development, related with recently discovered *essentially non-BPS* multimonopoles [49, 62, 18, 59, 60].

2.2 Singular SU(2) monopole with charge $g = ng_0$

As we attempt to construct in a non-Abelian theory a system of several (anti)monopoles, we run into a problem connected with the self-coupling of the gauge fields and non-linearity of the field equations. A simple superposition of the fields of two or more monopoles is no longer a stationary point of the action. However, as we saw in the previous section, in the Yang-Mills-Higgs non-Abelian model the gauge invariant electromagnetic field strength tensor (24) is defined as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - \frac{1}{e}\varepsilon_{abc}\hat{\phi}^a\partial_\mu\hat{\phi}^b\partial_\nu\hat{\phi}^c, \tag{53}$$

where the abelian gauge potential is projected out as $A_\mu = A_\mu^a\hat{\phi}^a$. Thus, we can perform a singular gauge transformations which rotates this configuration to an *Abelian gauge* where the scalar field is constant: $\tilde{\phi}^a = a\delta_{a3}$ and the gauge field has only one isotopic component, $\tilde{A}_k^3$. In such a gauge the second term in the definition of the electromagnetic field strength tensor (53) transforms into the singular field of the Dirac string and the magnetic charge is entirely associated with the topology of the gauge field [8].

Recall that gauge transformations, which connect the 'hedgehog' and Abelian gauges, are singular and we have to be very careful dealing with them. However, there is an obvious advantage in working in the Abelian gauge: here the gauge potentials are additive and the field equations are linear. That is why the authors of the paper [8] suggested to implement the following program of construction of a multi-monopole configuration: (i) start with an Abelian gauge, (ii) suppose that the gauge potential $\tilde{A}_k^3$ is a simple sum of a few singular Dirac monopoles embedded into the $SU(2)$ gauge group and then try to define a gauge transformation, which

1. Removes the singularity of the potential $\tilde{A}_k^3$ in the string gauge;

2. Provides proper asymptotic behaviour of the fields in the Higgs vacuum: the scalar field must smoothly tend to the vacuum value $|\phi| = a$, while the gauge potential must vanish as $1/r$.

Let us try to implement this program to construct a possible generalisation of the 't Hooft–Polyakov solution (34) to the case of non-minimal magnetic charge $g = ng_0 = 4\pi n/e$ [39, 40, 41]. We consider a 'fat' Dirac potential embedded into the $SU(2)$ model. Thus, we start with the Abelian gauge where the electromagnetic subgroup corresponds to rotation about the third axis of isospace:

$$\tilde{A}_k(\mathbf{r}) = \frac{n}{er}\frac{1 - \cos\theta}{\sin\theta}T_3\hat{\varphi}_k \ . \tag{54}$$

In the following discussion we will consider the fundamental representation of the group, $T_a = \frac{\sigma_a}{2}$.

Now we can make use of an analogy with the gauge transformation which rotates the Dirac potential to the non-Abelian Wu–Yang singular potential:

$$A_k = \frac{1}{2}A_k^a\sigma^a = U^{-1}\tilde{A}_kU - \frac{i}{e}U^{-1}\partial_kU, \qquad \phi^a = U\tilde{\phi}^aU^{-1} \ , \tag{55}$$

where

$$U(\theta, \varphi) = e^{-i(\sigma_k \hat{\varphi}_k^{(n)})\theta/2} = \begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2}e^{-in\varphi} \\ \sin\frac{\theta}{2}e^{in\varphi} & \cos\frac{\theta}{2} \end{pmatrix} . \tag{56}$$

Here the n-fold rotation in azimuthal angle φ is needed to balance the singular part of the Abelian potential (54), $\hat{\boldsymbol{\varphi}}^{(n)} = -\hat{\mathbf{e}}_1 \sin n\varphi + \hat{\mathbf{e}}_2 \cos n\varphi$.

To write the rotated potential in a compact form we define the $su(2)$ matrices $\tau_r^{(n)} = (\hat{\mathbf{r}}^{(n)} \cdot \boldsymbol{\sigma})$, $\tau_\theta^{(n)} = (\hat{\boldsymbol{\theta}}^{(n)} \cdot \boldsymbol{\sigma})$, and $\tau_\varphi^{(n)} = (\hat{\boldsymbol{\varphi}}^{(n)} \cdot \boldsymbol{\sigma})$. In this notation we obtain

$$A_k = \frac{1}{2er}\left(\tau_\varphi^{(n)}\hat{\theta}_k - n\tau_\theta^{(n)}\hat{\varphi}_k\right); \qquad \phi^a = a\hat{r}_a^{(n)} . \tag{57}$$

As one could expect, when rotated into the 'hedgehog' gauge, this configuration is spherically symmetric and the corresponding magnetic field

$$B_k = n\frac{r_k}{er^3} \tag{58}$$

is exactly the field of a static magnetic monopole with charge $g = 4\pi n/e$ at the origin. One can prove that for the configuration (57) the condition $D_n\phi^a = 0$ holds.

Note that the potential (57) is not a naive generalisation of the Wu-Yang potential

$$A_n = n\,\varepsilon_{amn}\frac{r_m}{r^2}\,,\frac{\sigma^a}{2} \tag{59}$$

which was considered in [42]. The configuration (59) with $n = 2$ has some amusing properties: The corresponding field strength tensor vanishes identically since the commutator term precisely cancel the derivative terms. Indeed, such a potential is a pure gauge:

$$A_n = iU^{-1}\partial_n U, \quad \text{where} \quad U = i\sigma^a\hat{r}^a$$

unlike the potential (57). Thus, this configuration is an unstable deformation of the topologically trival sector. We shall consider such deformations below.

Recall that the Wu–Yang configuration which was constructed via the gauge rotation of the embedded Dirac potential, is only the asymptotic limit of the 't Hooft–Polyakov solution (34) at $r \to \infty$. Unlike the former, the latter corresponds to finite energy of the configuration. It is easy to see that the configuration (57) is singular at the origin as well.

One can try to exploit an analogy with the 't Hooft–Polyakov ansatz, i.e., modify the asymptotic form of the fields (57) by including shape functions $H(r)$ and $K(r)$, respectively [39, 41]:

$$A_k = \frac{K(r)}{2er}\left(\tau_\varphi^{(n)}\hat{\theta}_k - n\tau_\theta^{(n)}\hat{\varphi}_k\right); \qquad \phi^a = aH(r)\hat{r}_a^{(n)} . \tag{60}$$

However, substitution of this ansatz into the field equations of the Yang–Mills–Higgs system (8) for $|n| \geq 2$ leads to a contradiction with the assumption that a regular solution of the form (60) could exist. Thus, we need to introduce more profile functions to obtain a smooth non-spherically symmetric solution of the field equations. However even that configuration

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will be unstable. Here, we can see a manifestation of the very general Lubkin theorem [44] (see, for example, the discussion in the Coleman lectures [43] and Nahm review in [38]). According to this theorem, there is a unique spherically symmetric monopole in the $SU(2)$ model (1) with minimal magnetic charge. Both analytical and numerical calculations have proved this conclusion [45, 46]. Therefore, the configuration, which has the asymptotic form (60), is a saddle point of the energy functional and it decays into a system of a few separated single monopoles and antimonopoles with total charge $n = n_+ - n_-$.

Indeed, there are axially symmetric saddle point solutions of the model (1) which approach the form (57) on the spacial asymptotic [55]. We review these monopoles below, in section 2.4.

2.3 Magnetic dipole

Let us continue our attempts to construct multi-monopole configurations. We consider now a magnetic dipole: a monopole–anti-monopole pair located on the z axis at the points $(0, 0, \pm L)$ with both strings directed along the positive z axis as in figure 2. A simple addition of the corresponding two singular Dirac potentials yields

$$A_k(\mathbf{r}) = \frac{1}{2e} \left(\frac{1 - \cos\theta_1}{r_1 \sin\theta_1} - \frac{1 - \cos\theta_2}{r_2 \sin\theta_2} \right) \sigma_3 \, \hat{\varphi}_k \; . \tag{61}$$
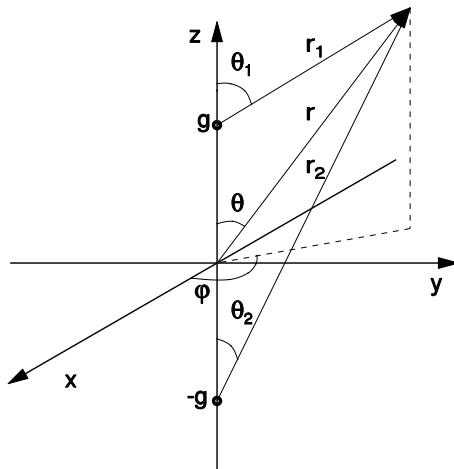
Figure 2: Magnetic dipole configuration from afar.

It was noted by Coleman that a crucial feature of this construction is that the Dirac strings of both monopoles lie along the same axis [43]. That allows to define a gauge transformation which removes the singularity of the potential (61) embedded into the $SU(2)$ group [8, 39]. Let us exploit the analogy with the transition of a monopole embedded into $SU(2)$, from the singular Dirac monopole in the Abelian gauge to the Wu–Yang non-Abelian monopole in the 'hedgehog' gauge. We see that the gauge transformation which

would remove the string singularity must rotate a unit isovector $\hat{\mathbf{n}}$ associated with the direction of the string, about the third isospace axis by an angle 4π. However, this vector now originates from the points $|z| = L$. Therefore, the gauge transformation we are looking for, must become an element of unity for $|z| > L$. A proper choice is [8, 39]

$$U(\theta_1 - \theta_2, \varphi) = e^{-\frac{i}{2}\varphi\sigma_3} e^{\frac{i}{2}(\theta_1-\theta_2)\sigma_2} e^{\frac{i}{2}\varphi\sigma_3} = \begin{pmatrix} \cos\frac{\theta_1-\theta_2}{2} & -\sin\frac{\theta_1-\theta_2}{2}e^{-i\phi} \\ \sin\frac{\theta_1-\theta_2}{2}e^{i\phi} & \cos\frac{\theta_1-\theta_2}{2} \end{pmatrix}. \tag{62}$$

A simple calculation shows that the Higgs field after rotation into the 'hedgehog' gauge is

$$\tilde{\phi}^a = U\phi^a U^{-1} = a(\sin(\theta_1 - \theta_2)\cos\varphi, \sin(\theta_1 - \theta_2)\sin\varphi, \cos(\theta_1 - \theta_2)) \tag{63}$$

One can prove that the gauge transformation (62) removes the singularity of the embedded potential (61).

A generalization of this procedure allows also to rotate into a non-singular gauge some other configurations: a monopole–anti-monopole pair connected with a Dirac string, a monopole–monopole pair, or a system of a few monopoles lying along a line [39]. Moreover, it is possible to generalize this procedure to the case of an arbitrary gauge group, for example, an $SU(3)$ magnetic dipole was considered in [47]. Recently we make use of the Abelian gauge to analyse interaction between two different well separated non-BPS $SU(3)$ fundamental monopoles; this is an amusing situation of attraction between two positive or negative charges [48]. The only restriction is that all the strings must be directed along the same line; otherwise it is impossible to remove all the singularities of the multi-monopole potential by making use of a singular gauge transformation [43]. In other words, such a multi-monopole configuration after rotation to a 'hedgehog' gauge is described as a system of a few monopoles having identical orientation in isotopic space. Obviously, that is not the case for an arbitrary multi-monopole system we are looking for.

Another inconsistency of the description above is that these expressions have a restricted domain of applicability.

Indeed, there was a hidden contradiction in our discussion above. Actually, so far we are dealing with point-like monopoles because our configuration is just a generalization of the non-Abelian Wu–Yang potential. The regular 't Hooft–Polyakov solution coincides with it only asymptotically. Thus, there is still a question of the inner structure of the monopoles, or in other words, the problem of finding a solution which would make the Higgs field vanish at some points, which are associated with the positions of the monopoles.

The contradiction is that, on one hand we suppose that each monopole is characterized by a topological charge connected with the spatial asymptotic of the scalar field. On the other hand, the definition of a magnetic dipole implies that the monopoles are separated by a finite distance $2L$, and moreover, $L \ll r$. A proper approximation would therefore not be a magnetic dipole, but rather a monopole–anti-monopole pair separated by a distance which is very large compared to size of the core. Indeed, there is a smooth, finite energy magnetic dipole solution to the model (1) where two zeros of the Higgs field are relatively close to each other [49, 58].

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2.4 Rebbi-Rossi multimonopoles, monopole-antimonopole chains and closed vortices

Discussion of the magnetic dipole 'from afar', yields some clue to the structure of the solution we sought for. The configuration space of YMH theory consists of sectors characterized by the topological charge of the Higgs field. While the unit charge 't Hooft - Polyakov hedgehog solution (34) corresponds to a single covering of the vacuum manifols S^2_{vac} is by a single turn around the spatial boundary S^2, multimonopole configurations have to be characterised by n-fold covering of the vacuum manifold. This configuration is spherically symmetric. It was shown that $SU(2)$ monopoles with higher topological charge cannot be spherically symmetric [50] and posess at most axial symmetry [55, 16, 51, 52] or no rotational symmetry at all [17, 57].

In the BPS limit of vanishing Higgs potential spherically symmetric monopole and axially symmetric multimonopole solutions, which satisfy the first order Bogomol'nyi equations (41) are known analytically [16, 51, 52]. These solutions were constructed on the way which we outlined in section 2.1. For these solutions all zeros of the Higgs field are superimposed at a single point. Multimonopole solutions of the Bogomol'nyi equations which do not possess any rotational symmetry [56], have recently been constructed numerically by making use of the rational map ansatz [17]. In these solutions the zeros of the Higgs field are no longer all superimposed at a single point but are located at several isolated points [5].

The energy of the BPS solutions satisfies exactly the lower energy bound given by the topological charge. As shown by Taubes using infinite dimensional Morse theory, in each topological sector there exist in addition smooth, finite energy solutions of the second order field equations, which do not satisfy the Bogomol'nyi equations but only the second order field equations (8) [20]. Consequently, the energy of these solutions exceeds the Bogomol'nyi bound.

The simplest such solution resides in the sector with zero topological charge and corresponds to a saddlepoint of the energy functional [20]. It possesses axial symmetry, and the two zeros of its Higgs field are located symmetrically on the positive and negative z-axis. This solution corresponds to a monopole and antimonopole in static equilibrium [49, 58]. Such a configuration is a deformations of the topologically trivial sector.

Spherically symmetric non-Bogomol'nyi BPS $SU(N)$ monopoles were considered by Ioannidou and Sutcliffe [18]. Very recently we discovered new classical solutions, which are associated with monopole-antimonopole systems in $SU(2)$ YMH theory with the scalar field in the adjoint representation [59, 60, 61]. In these solutions, which generalizes both Rebbi-Rossi multimonopoles [55] and monopole-antimonopole pair solution [49, 58], the Higgs field vanishes either at some set of discrete isolated points or at rings. The latter configurations corresponds to the closed vortices while the former are (multi)-monopole-antimonopole bound systems. There are also a third class of solutions which corresponds

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[5]Static spherically symmetric $SU(N)$ non-Bogomol'nyi BPS monopoles were recently constructed for $N > 2$ [18].

to a single (multi) monopole bounded with a system of vortex rings centered around the symmetry axis. We review these configurations below.

2.4.1 Static axially symmetric ansatz

Since the Higgs field is taking values in $su(2)$ Lie algebra, we may consider a triplet of unit vectors

$$
\begin{aligned}
\hat{e}_r^{(n,m)} &= [\sin(m\theta)\cos(n\varphi), \sin(m\theta)\sin(n\varphi), \cos(m\theta)]; \\
\hat{e}_\theta^{(n,m)} &= [\cos(m\theta)\cos(n\varphi), \cos(m\theta)\sin(n\varphi), -\sin(m\theta)]; \\
\hat{e}_\varphi^{(n)} &= [-\sin(n\varphi), \cos(n\varphi), 0]
\end{aligned}
\tag{64}
$$

which describe both rotations in azimuthal angle and in polar angle as well.

Now we define the $su(2)$ matrices $\tau_r^{(n,m)}$, $\tau_\theta^{(n,m)}$, and $\tau_\varphi^{(n)}$ as a product of these vectors with the usual Pauli matrices $\tau^a = (\tau_x, \tau_y, \tau_z)$:

$$
\begin{aligned}
\tau_r^{(n,m)} &= \sin(m\theta)\tau_\rho^{(n)} + \cos(m\theta)\tau_z , \\
\tau_\theta^{(n,m)} &= \cos(m\theta)\tau_\rho^{(n)} - \sin(m\theta)\tau_z , \\
\tau_\varphi^{(n)} &= -\sin(n\varphi)\tau_x + \cos(n\varphi)\tau_y ,
\end{aligned}
$$

where $\tau_\rho^{(n)} = \cos(n\varphi)\tau_x + \sin(n\varphi)\tau_y$ and $\rho = \sqrt{x^2 + y^2} = r\sin\theta$.

We parametrize the gauge potential and the Higgs field by the static, purely magnetic Ansatz

$$
A_\mu dx^\mu = \left(\frac{K_1}{r}dr + (1-K_2)d\theta\right)\frac{\tau_\varphi^{(n)}}{2e} - n\sin\theta\left(K_3\frac{\tau_r^{(n,m)}}{2e} + (1-K_4)\frac{\tau_\theta^{(n,m)}}{2e}\right)d\varphi
\tag{65}
$$

$$
\Phi = \Phi_1\tau_r^{(n,m)} + \Phi_2\tau_\theta^{(n,m)} .
\tag{66}
$$

which generalises the spherically symmetric 't Hooft-Polyakov ansatz (34). The latter can be recovered if we impose the constrains $K_1 = K_3 = \Phi_2 = 0$, $K_2 = K_4 = K(\xi)$, $\Phi_1 = H(\xi)$.

We refer to the integers m and n in (64),(65),(66) as θ winding number and φ winding number, respectively. Indeed, as the unit vector (64) parametrized by the polar angle θ and asimuthal angle φ covers the sphere S_2 once, the fields defined by the Ansatz (65),(66) wind n and m times around z-axis and ρ-axis respectively.

There are some useful relations between the matrices $\tau_r^{(n,m)}$, $\tau_\theta^{(n,m)}$, and $\tau_\varphi^{(n)}$:

$$
\partial_\theta\tau_r^{(n,m)} = m\tau_\theta^{(n,m)}; \qquad \partial_\theta\tau_\theta^{(n,m)} = -m\tau_r^{(n,m)}; \qquad \partial_\varphi\tau_r^{(n,m)} = n\sin(m\theta)\tau_\varphi^{(n)}
$$

$$
\partial_\varphi\tau_\theta^{(n,m)} = n\cos(m\theta)\tau_\varphi^{(n)}; \qquad \partial_\varphi\tau_\varphi^{(n)} = -n[\sin(m\theta)\tau_r^{(n,m)} + \cos(m\theta)\tau_\theta^{(n,m)}];
\tag{67}
$$

$$
\tau_z = \cos(m\theta)\tau_r^{(n,m)} - \sin(m\theta)\tau_\theta^{(n,m)} .
$$

Profile functions $K_1 - K_4$ and Φ_1, Φ_2 depend on the coordinates r and θ only. Thus, this ansatz is axially symmetric in a sence that a spacial rotation around the z-axis can be

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compencated by an Abelian gauge transformation $U = \exp\{i\omega(r,\theta)\tau_\varphi^{(n)}/2\}$ which leaves the ansatz form-invariant.

To obtain a regular solution we make use of the $U(1)$ gauge symmetry. to fix the gauge [62]. We impose the condition

$$G_f = \frac{1}{r^2}\left(r\partial_r K_1 - \partial_\theta K_2\right) = 0 \ .$$

The gauge fixing term $L_\eta = \eta G_f^2$ must be added to the Lagrangian (1).

With this Ansatz the field strength tensor components become

$$F_{r\theta} = -\frac{1}{r}\left(\partial_\theta K_1 - r\partial_r K_2\right)\frac{\tau_\varphi^{(n)}}{2}$$

$$F_{r\varphi} = -\frac{n}{r}\sin\theta\left[\left(K_1\frac{\sin(m\theta)}{\sin\theta} + K_1(K_4 - 1) - r\partial_r K_3\right)\frac{\tau_r^{(n,m)}}{2}\right.$$
$$\left. + \left(K_1\frac{\cos(m\theta)}{\sin\theta} + K_1 K_3 + r\partial_r K_4\right)\frac{\tau_\theta^{(n,m)}}{2}\right];$$

$$F_{\theta\varphi} = n\sin\theta\left[\left((1 - K_2)\frac{\sin(m\theta)}{\sin\theta} + (1 - K_4)(K_2 + n - 1) - \partial_\theta K_3 - K_3\cot\theta\right)\frac{\tau_r^{(n,m)}}{2}\right.$$
$$\left. + \left((1 - K_2)\frac{\cos(m\theta)}{\sin\theta} - K_3(K_2 + n - 1) + \partial_\theta K_4 - (1 - K_4)\cot\theta\right)\frac{\tau_\theta^{(n,m)}}{2}\right] \ .$$
$$(68)$$

and the components of covariant derivative of the Higgs field

$$D_r\Phi = \frac{1}{r}\left([r\partial_r\Phi_1 + K_1\Phi_2]\,\tau_r^{(n,m)} + [r\partial_r\Phi_2 - K_1\Phi_1]\,\tau_\theta^{(n,m)}\right);$$
$$D_\theta\Phi = [\partial_\theta\Phi_1 - K_2\Phi_2]\,\tau_r^{(n,m)} + [\partial_\theta\Phi_2 + K_2\Phi_1]\,\tau_\theta^{(n,m)};$$
$$D_\varphi\Phi = m\left[\sin\theta(K_3\Phi_2 + K_4\Phi_1) + \cos\theta\Phi_2\right]\tau_\varphi^{(n)} \ .$$
$$(69)$$

Variation of the Lagrangian (1) with respect to the profile functions yelds a system of six second order non-linear partial differential equations in the coordinates r and θ.

2.4.2 Boundary conditions

To obtain regular solutions with finite energy density and correct asymptotic behavour we impose the boundary conditions. Regularity at the origin requires

$$K_1(0,\theta) = 0 \ , \qquad K_2(0,\theta) = 1 \ , \qquad K_3(0,\theta) = 0 \ , \qquad K_4(0,\theta) = 1 \ ,$$

$$\sin(m\theta)\Phi_1(0,\theta) + \cos(m\theta)\Phi_2(0,\theta) = 0 \ ,$$
$$\partial_r\left[\cos(m\theta)\Phi_1(r,\theta) - \sin(m\theta)\Phi_2(r,\theta)\right]\big|_{r=0} = 0$$

that is $\Phi_\rho(0, \theta) = 0$.

To obtain the boundary conditions at infinity we require that solutions in the vacuum sector $(m = 2k)$ tend to a gauge transformed trivial solution,

$$\Phi \longrightarrow U\tau_z U^\dagger \ , \quad A_\mu \longrightarrow i\partial_\mu U U^\dagger \ ,$$

and the solutions in the topological charge n sector $(m = 2k + 1)$ to tend to

$$\Phi \longrightarrow U\Phi_\infty^{(1,n)} U^\dagger \ , \quad A_\mu \longrightarrow U A_{\mu\infty}^{(1,n)} U^\dagger + i\partial_\mu U U^\dagger \ ,$$

where

$$\Phi_\infty^{(1,n)} = \tau_r^{(1,n)} \ , \quad A_{\mu\infty}^{(1,n)} dx^\mu = \frac{\tau_\varphi^{(n)}}{2} d\theta - n\sin\theta \frac{\tau_\theta^{(1,n)}}{2} d\varphi$$

is the asymptotic solution of a charge n multimonopole, and $SU(2)$ matrix $U = \exp\{-ik\theta\tau_\varphi^{(n)}\}$, both for even and odd m. Consequently, solutions with even m have vanishing magnetic charge, whereas solutions with odd m possess magnetic charge n.

In terms of the functions $K_1 - K_4$, Φ_1, Φ_2 these boundary conditions read

$$K_1 \longrightarrow 0 \ , \quad K_2 \longrightarrow 1 - m \ , \tag{70}$$

$$K_3 \longrightarrow \frac{\cos\theta - \cos(m\theta)}{\sin\theta} \quad m \text{ odd} \ , \quad K_3 \longrightarrow \frac{1 - \cos(m\theta)}{\sin\theta} \quad m \text{ even} \ , \tag{71}$$

$$K_4 \longrightarrow 1 - \frac{\sin(m\theta)}{\sin\theta} \ , \tag{72}$$

$$\Phi_1 \longrightarrow 1 \ , \quad \Phi_2 \longrightarrow 0 \ . \tag{73}$$

Regularity on the z-axis, finally, requires

$$K_1 = K_3 = \Phi_2 = 0 \ , \quad \partial_\theta K_2 = \partial_\theta K_4 = \partial_\theta \Phi_1 = 0 \ ,$$

for $\theta = 0$ and $\theta = \pi$.

Subject of the above boundary conditions, we constructed numerically solutions with $1 \leq m \leq 6$, $1 \leq n \leq 6$ and several values of the Higgs selfcoupling constant λ.

2.4.3 Properties of the solutions

We give here a description of the general properties of the solutions. Note that asymptotic behavior of the profile functions allows to check the equivalence between the topological

charge Q and magnetic charge g. Indeed,

$$
\begin{aligned}
Q &= \frac{1}{8\pi} \int\limits_{S_2} d^2\xi \ \varepsilon_{\alpha\beta}\varepsilon_{abc}\phi^a\partial_\alpha\phi^b\partial_\beta\phi^c = \\
&= \frac{2nm}{8\pi} \int d\theta d\varphi \ [H_1\sin(m\theta) + H_2\cos(m\theta)] = \frac{n}{2}\left[1 - (-1)^m\right]; \\
g &= \frac{1}{4\pi} \int \mathrm{Tr}\,\varepsilon_{ijk}\left(F_{ij}D_k\hat{\Phi}\right) d^3r = \frac{1}{2\pi} \int\limits_{S_2} d\theta d\varphi \ \mathrm{Tr}(F_{\theta\varphi}\Phi) \\
&= \frac{n}{2} \int d\theta \left(\sin(m\theta) - \partial_\theta(\sin\theta K_3)\right) = \frac{n}{2}\left[1 - (-1)^m\right]
\end{aligned}
\tag{74}
$$

where we used the definitions (28), (39) and substitute the asymptotic behavior of the provile functions (71),(72) (73). Thus, the configurations given by the axially symmetric Ansatz (65),(66) are either deformations of the topologically trivial sector (e.g. monopole-antimonopole pair, $n = 1$, $m = 2$) or deformations of core of charge n Rebbi-Rossi multi-monopoles (e.g. separation of two zeros of the Higgs field of the charge 2 monopole).

2.4.4   2d Poincare index of the vector field

The axial symmetry of the ansatz (65),(66) means that we can choose any value of the azimuthal angle, for example $\varphi = 0$, that is consider the behaviour of the fields on the xz plane. Then classification of the solutions can be constructed by making use of the *2d Poincare index* of the Higgs field on that plane which supplements 3d topological characteristic (32) [63].

By analogy with (32), 2d index is defined by consideration of a smooth two-dimensional vector field $\vec{v}(x) = (v_1, v_2)$ on a compact space X with an isolated zero at x_0. The function $f(x) = \frac{\vec{v}(x)}{|v(x)|}$ maps a small circle S^1 centered at x_0 into the unit sphere S^1_ϕ. The degree of this mapping is called the Poincare index i_{x_0} of vector field at the zero x_0. If the circle is parametrized by an angle $\alpha \in [0 : 2\pi]$, the index is

$$
i_{x_0} = \frac{1}{2\pi} \int\limits_{S_1} \frac{\vec{v} \wedge d\vec{v}}{|v|^2} = \frac{1}{2\pi} \int d\alpha \ \frac{v_1 d_\alpha v_2 - v_2 d_\alpha v_1}{v_1^2 + v_2^2}
\tag{75}
$$

Note that different orientation of the vector field at the point $\alpha = 0$ on the small circle S_1 yields configurations which are labeled by the same index (see Fig. 3). Any perturbation of the contour without crossing a node of the 2-dimensional vector field does not change the index.

Let us consider a closed contour near spacial boundary of a plane which encircles some set of isolated zeros of a two-dimensional vector field [6]

---

[6]This corresponds to an infinitesimal contour around the pole of sphere S_2 which can be obtained by compactification of that plane. Then the vector field on S_2 is pointed outward at all boundary points and the sum of all indices of zeros of such a vector field is equal to the Euler number $\chi(S_2) = 2$, a topological invariant of X, which does not depend on particular choice of vector field. This is the original statement of the Poincare-Hopf theorem.
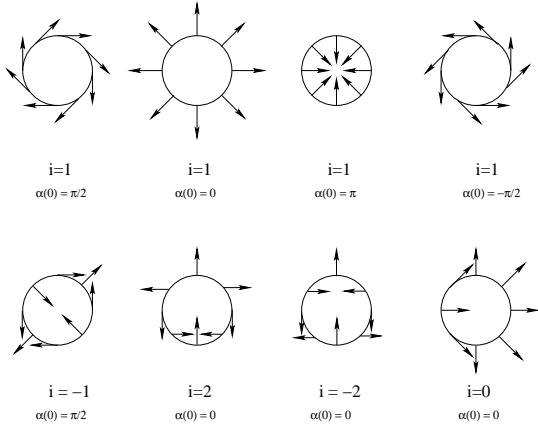
Figure 3: Indices of two dimensional vector field.

Mapping of a vector field $\vec{v}(x)$ at the spacial infinity into a unit circle yields the index i_∞. This index will not change upon a perturbation of the field, as long as no node crosses the contour during perturbation. The Poincare-Hopf theorem then can be formulated as a statement that the index of that field with respect to a contour near infinity is equal to the sum of the indices computed locally around each node of the field:

$$i_\infty = \sum_k i(x_0^{(k)})$$

In the case under consideration we consider nodes of the Higgs field on the xz plane. The boundary condition (73) means that $\Phi(r,\theta) \to \tau_r^{(n,m)}$ as $r \to \infty$. Thus, $\Phi_y = 0$ and coresponding components of two-dimensional vector field are $v_1 \equiv \Phi_z = \cos(m\theta), v_2 \equiv \Phi_x = \sin(m\theta)$ as $x > 0$ and $v_1 \equiv \Phi_z = \cos(m\theta), v_2 \equiv \Phi_x = (-1)^n \sin(m\theta)$ as $x < 0$ respectively. Then calculation of the index i_∞ according to eq. (75) yields

$$i_\infty = \frac{m}{2}\left[1 - (-1)^n\right] \tag{76}$$

This is a 2-dimensional topological characteristics of the axially symmetric solutions which is complementary to the 3-dimensional invariant (32). Note that it is "dual" to the topological charge (74) in a sence that permutation $n \rightleftharpoons m$ yields $i_\infty \rightleftharpoons Q$. However, the index i_∞ is not an invariant of the transformations between homotopically related configurations within the same topological sector. Corresponding deformations of 3-dimensional vector field may be related with vanishing of the projection of the field on xz plane on the circle at the spacial infinity. The crossing of this node yields jump of the 2-dimensional index i_∞ by some integer value.

2.4.5 m-chains

The solution of the first type are m-chains constructed in [59, 60]. They are characterized by θ winding number $m > 1$ and φ winding number $n = 1, 2$.

Let us consider $n = 1$ configurations first. These m-chains possess m nodes of the Higgs field on the z-axis. Due to reflection symmetry, each node on the negative z-axis
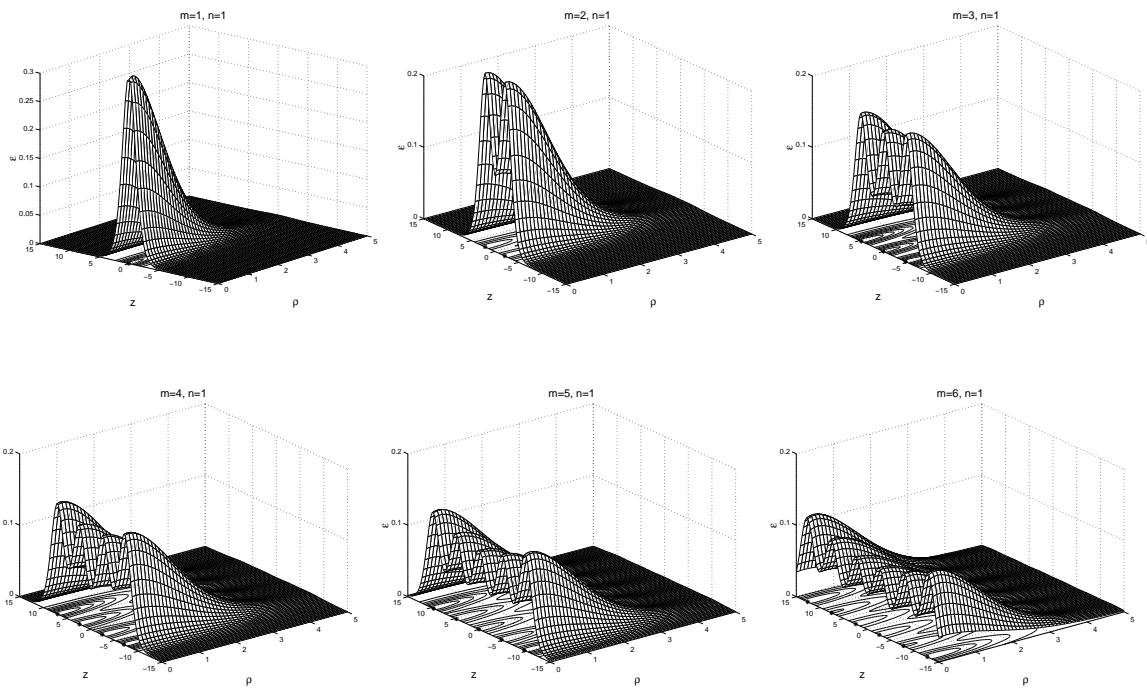
Figure 4: The rescaled energy density $E(\rho, z)$ is shown for the monopole-antimonopole chains with winding numbers $n = 1, m = 1, \ldots 6$ at $\lambda = 0$.

corresponds to a node on the positive z-axis. The nodes of the Higgs field $x_0(k)$ are associated with the location of the monopoles and antimonopoles and all of them are characterized by the index $i_{(}x_0^{(k)}) = 1$ (cf Fig. 2). The index i_∞ of these m-chains is obviously equal to the winding number m whereas the topological charge is either unity (for odd m) or zero (for even m).

Indeed, for odd m ($m = 2k + 1$) the Higgs field possesses k nodes on the positive z-axis and one node at the origin. The node at the origin corresponds to a monopole if k is even and to an antimonopole if k is odd. For even m ($m = 2k$) there is no node of the Higgs field at the origin.

The $m = 1$ solution is the spherically symmetric 't Hooft-Polyakov monopole which we discussed in the first part of our review. The $m = 3$ (M-A-M) and $m = 5$ (M-A-M-A-M) chains represent saddlepoints with unit topological charge. The $m = 2$ chain is identical to the monopole-antimonopole (M-A) pair discussed in [49, 58]. The M-A pair as well as the $m = 4$ (M-A-M-A) and $m = 6$ (M-A-M-A-M-A) chains form saddlepoints in the vacuum sector.

In Fig. 4 we present the dimensionless energy density and nodes of the Higgs field for the $n = 1$ solutions with θ winding number $m = 1, \ldots, 6$. The energy density of the m-chain possesses m maxima on the z-axis, and decreases with increasing ρ. The locations of the maxima are close to the nodes of the Higgs field. For a given m the maxima are of similar magnitude, but their height decreases with increasing m. Increasing of λ makes these maxima sharper and decreases the distance between the locations of the monopoles. We observe that for a given λ the distances between the corresponding nodes increase with increasing m.

We observe that the energy $E^{(m)}$ of an m-chain is always smaller than the energy of m

single monopoles or antimonopoles (with infinite separation between them), i. e. $E^{(m)} < E_\infty = 4\pi\eta m$. On the other hand $E^{(m)}$ exceeds the minimal energy bound given by the Bogolmol'ny limit $E_{\min}/4\pi\eta = 0$ for even m, and $E_{\min}/4\pi\eta = 1$ for odd m. We observe an (almost) linear dependence of the energy $E^{(m)}$ on m. This can be modelled by taking into account only the energy of m single (infinitely separated) monopoles and the next-neighbour interaction between monopoles and antimonopoles on the chain. Defining the interaction energy as the binding energy of the monopole-antimonopole pair,

$$\Delta E = 2\,(4\pi\eta) - E^{(2)} \ ,$$

we obtain as energy estimate for the m-chain

$$E_{\text{est}}^{(m)}/4\pi\eta = m + (m-1)\Delta E \ .$$
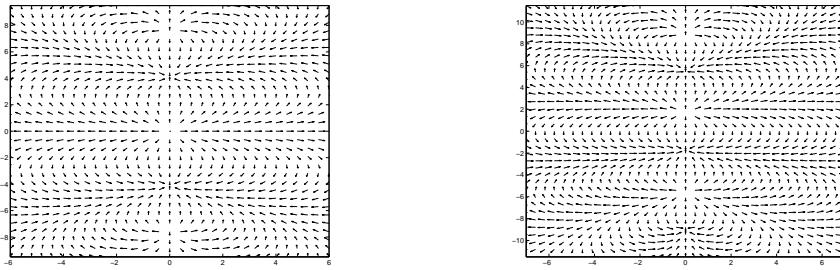
Figure 5: 2d Higgs field of monopole-antimonopole chains ($m = 5, n = 1$ and $m = 6, n = 1$) All the nodes correspond to the unit index.

We interpret the m-chains as equilibrium states of m non-BPS monopoles and anti-monopoles. As shown long ago [19], the force between BPS monopoles is given by twice the Coloumb force when the charges are unequal, and vanishes when the charges are equal, provided the monopoles well separated. Thus, monopoles and antimonopoles can only be in static equilibrium, if they are close enough to experience a repulsive force that counteracts the attraction. Indeed, is the separation between monopoles is not too large, both photon and scalar particles remain massive, the vector bosons $A_\mu^\pm$ are "defrosted" and all the particles from the spectrum contribute to the short-range Yukawa-type interactions.

In other words, m-chains are essentially non-BPS solutions. To see this in another way let us consider the limit $\lambda = 0$. Then the energy of these 'non-Bogomol'nyi BPS monopoles' can be written in the form (40):

$$E = \int \left\{ \frac{1}{4}\text{Tr}\left((\varepsilon_{ijk}F_{ij} \pm D_k\Phi)^2\right) \mp \frac{1}{2}\varepsilon_{ijk}\text{Tr}\left(F_{ij}D_k\Phi\right) \right\} d^3r \ . \tag{77}$$

The second term is proportional to the topological charge and vanishes when m is even. The first term is just the integral of the square of the Bogomol'nyi equations. Thus, for even m the energy is a measure for the deviation of the solution from selfduality.

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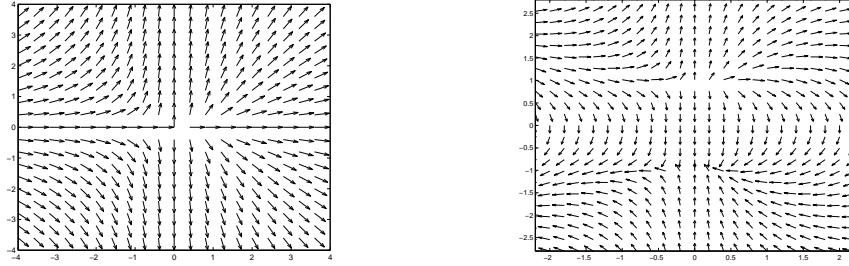
Figure 6: 2d Higgs field of a single 2-monopole and 2-monopole—2-antimonopole pair $(m = 1, n = 2$ and $m = 2, n = 2)$ The double nodes correspond to zero indices.

Let us now consider chains consisting of multimonopoles with winding number $n = 2$. Identifying the locations of the Higgs zeros on the symmetry axis with the locations of the monopoles and antimonopoles, we observe that when each pole carries charge $n = 2$, the zeros form pairs, where the distance between the monopole and the antimonopole of a pair is less than the distance to the neighboring monopole or antimonopole, belonging to the next pair.

We observe furthermore, that the equilibrium distance of the monopole-antimonopole pair composed of $n = 2$ multimonopoles is smaller than the equilibrium distance of the monopole-antimonopole pair composed of single monopoles. Thus the higher attraction between the poles of a pair with charge $n = 2$ is balanced by the repulsion only at a smaller equilibrium distance.

The difference from the chains composed of pairs of monopole-antimonopole with winding number $n = 1$ we considered above, is that the index i_∞ of the m-chains composed of $n = 2$ multimonopoles is zero. Moreover, the index of each double node of the Higgs field, which is compatible with the symmetry conditions, is also zero. Another difference is that for $n = 2$ chains maxima of the energy density are no longer coincide with (double) zeros of the Higgs field. The latter still are placed on the z axis (cf. Fig. 5).

2.4.6 Closed vortices

The boundary conditions imposed at the origin on the Higgs field (66) means that $\Phi_\rho(0, \theta) = 0$ and

$$\Phi(0, \theta) = \Phi_z(0, \theta)\tau_z^{(n,m)}$$

Thus, the scalar field can either vanish there (for an odd θ winding number m), or to be directed along z axis (for an even m). In the former case there is a single n-monopole placed at the origin whereas the latter configuration in the limit of very large scalar coupling approaches the vacuum expectation value not only on the spacial boundary but also in the vicinity of the origin. Then the solutions with different winding number n link the trivial configuration $\Phi(0, \theta) = (0, 0, \Phi_z(0, \theta))$ and its gauge rotated on the spacial infinity. For $n = 1, 2$ these solutions are monopole-antimonopole chains we discussed above.
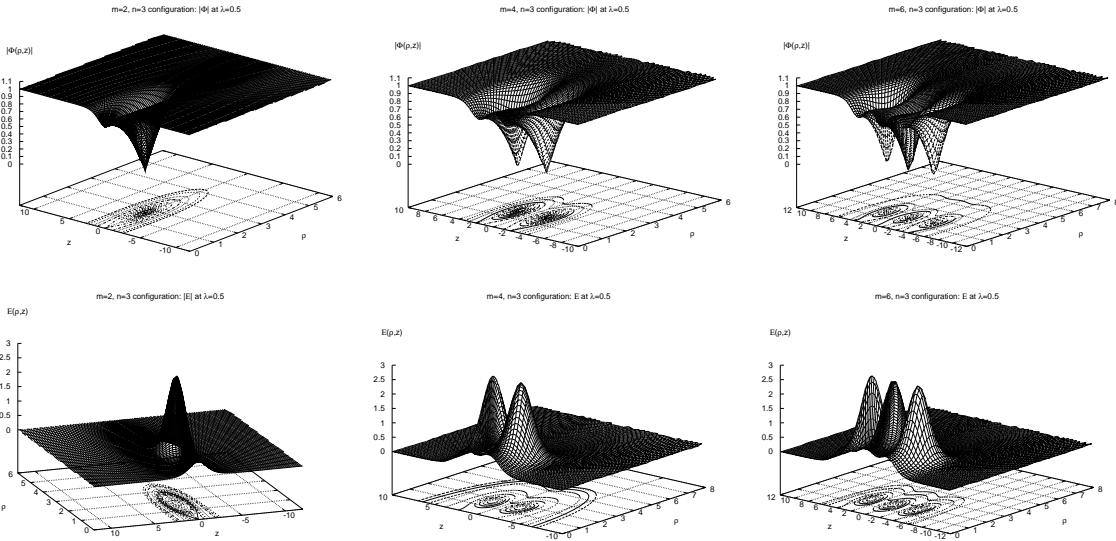
Figure 7: The rescaled energy density $E(\rho, z)$ and the modulus of the Higgs field $|\Phi(\rho, z)|$ are shown for the circular vortex solutions $m = 2, n = 3$ (single vortex), $m = 4, n = 3$ (double vortex) and $m = 6, n = 3$ (triple vortex) at $\lambda = 0.5$.

The situation changes dramatically if φ winding number $n > 2$. The solution of the second type, which appears in that case are not multimonopole chains but sustems of vortex rings, in the BPS limit either bounded with a single n-monopole placed at the origin, or without it [60]. One could expect that when charge of poles is increasing further beyond $n = 2$ the chain solutions consisting of multimonopoles with winding number n should exist, the monopoles and antimonopoles of the pairs should approach each other further, settling at a still smaller equilibrium distance.

Constructing solutions with charge $n = 3$ in the BPS limit ($\lambda = 0$) however, we do not find chains at all. Now there is no longer sufficient repulsion to balance the strong attraction between the 3-monopoles and 3-antimonopoles. Instead of chains, we now observe solutions with vortex rings, where the Higgs field vanishes on closed rings around the symmetry axis.

Such a situation is not a novelty, for examply the rings of zeros of a Higgs field are observed in Alice electrodynamics [65]. It is also known that closed knotted vortices arise in theories with a non-trivial Hopf number [66]. However the closed vortex solutions of Yang-Mills-Higgs theory were previously unknown.

To better understand these findings let us consider unphysical intermediate configurations, where we allow the φ winding number n to continuously vary between the physical integer values. Beginning with the simplest such solution, the $m = 2$ solution, we observe, that the zeros of the solution with winding number n continue to approach each other when n is increased beyond 2, until they merge at the origin. Here the pole and antipole do not annihilate, however. We conclude, that this is not allowed by the imposed symmetries and boundary conditions. Instead the Higgs zero changes its character completely, when n is further increased. It turns into a ring with increasing radius for increasing n. The physical

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3-monopole-3-antimonopole solution in the BPS limit then has a single ring of zeros of the Higgs field and no point zeros. The nodes of the Higgs field on the xz plane have local indices $+1$, thus the $2d$ index computed on the spacial boundary in $i_\infty = 2$.

Considering the magnetic moment of the $m = 2$ solutions, we observe, that it is (roughly) proportional to n. The pair of poles on the z-axis for $n = 2$ clearly gives rise to the magnetic dipole moment of a physical dipole. The ring of zeros also corresponds to a magnetic dipole field, which however looks like the field of a ring of mathematical dipoles. This corresponds to the simple picture that the positive and negative charges have merged but not annihilated, and then spread out on a ring. Thus, we can identify the $m = 2$, $n = 3$ solution with a closed vortex configuration.

While the dipole moment of monopole-antimonopole chains with an equal number of monopoles and antimonopoles has its origin in the magnetic charges of the configuration [58, 59], the dipole moment of the closed vortices can be associated with loops of electric currents in analogy with the case of the sphaleron dipole moment [64]. Indeed, the electromagnetic current can be defined as

$$j_i^{em} = \partial_k F_{ik} \ ,$$

where F_{ik} is the 't Hooft field strength thensor (24) and there are closed rings of currents inside of the core of the configuration [61].

Other solutions with even θ winding number reside in the vacuum sector as well. For $m = 2k > 2$ solutions with zero scalar coupling it is now clear how they evolve, when the φ winding number is increased beyond $n = 2$. Starting from k pairs of physical dipoles, the pairs merge and form k vortex rings, which carry the dipole strength of the solutions.

2.4.7 $n > 2$ and odd m: Vortices bounded with monopole

The solutions with odd θ winding number have an isolated zero of the Higgs field at the origin, thus their reside in the topological sector with charge n. For $m = 4k + 1$ the situation is somewhat similar to the above. Here a single n-monopole remains at the origin, whereas all other zeros form pairs, which for $n > 2$ approach each other, merge and form $2k$ rings carrying dipole strength. Since, however, a dipole on the positive axis and its respective counterpart on the negative axis have opposite orientation, their contributions cancel in the total magnetic moment. Thus the magnetic moment remains zero, as it must, because of the symmetry of the ansatz [59, 61]. Non-zero scalar coupling does not change the situation but the picks of the energy density are getting sharper.

For $m = 4k - 1$, on the other hand, the situation is more complicated. Let us consider the simplest case $m = 3$ in the limit $\lambda = 0$. Again, we consider unphysical configurations with winding number n continuously vary between $n = 2$ chain solution and $n = 3$ configuration. The difference from the case of an even m is that in the initial state there are 3 poles on the z-axis, which cannot form pairs, such that all zeros belong to a pair, symmetrically located around the origin, thus the dipole moment of that configuration is zero. For $m = 3$, we observe in the BPS limit, that two vortices appear in the charge $n = 3$ solution, emerging from the upper and lower unpaired zero, respectively, carrying opposite
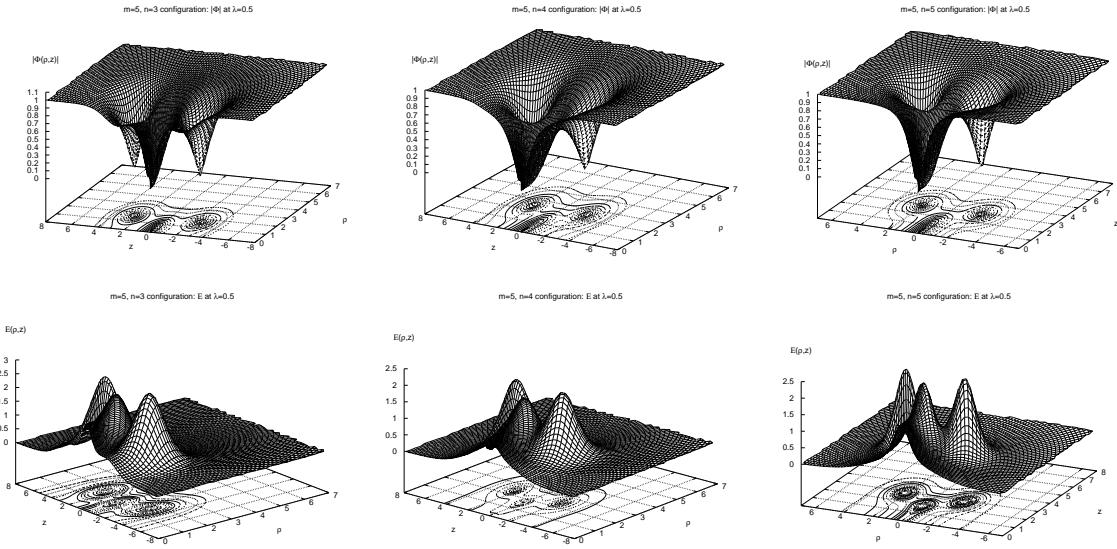
Figure 8: The rescaled energy density $E(\rho, z)$ and the modulus of the Higgs field $|\Phi(\rho, z)|$ are shown for the monopole-double vortex solution with winding numbers $m = 5, n = 3, 4, 5$ at $\lambda = 0.5$

dipole strength. Thus, the local $2d$ index at the origin is -1 whereas other four indices are 1, thus $i_\infty = 3$. Increasing of scalar coupling decreases the radius of the vortices as well the distance between then. However both rings remain individual.

This qualitative sketch of the properties of the new non-BPS axially symmetric multimonopoles and closed vortices is, of course, very superficial. We refer the reader to the papers [58, 59, 60, 61]. Nevertheless, there are many questions which remains to be studied. This work is currently underway.

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